13

Theories of Elastic Failure

13.1. INTRODUCTION

A machine element or a structural member may fail to perform its function under a given loading due to many reasons, the most important out of which is the yielding. Beyond yield point permanent change in the machine element under load occurs, which renders it unfit to perform its assigned function. Due to this reason, *yielding* is considered as the most important failure criterion.

Yield point under uniaxial loading can be obtained by testing the material specimen in laboratory. Then it can be said that any machine element when loaded in uniaxial direction will fail at the same value of yield stress as was determined in laboratory. However, what happens under multiaxial loading, *i.e.* the compound stress system is not so easy to comprehend or calculate. As a matter of fact, it cannot be said with cent per cent surety what exactly it is that causes failure ; whether it is normal stress, shear stress or any other parameter. Of course, various theories have been put forward and compared with experimental results to prove their validity or otherwise. So far no single theory has been developed which agrees with experimental results in case of all materials and all conditions of loading. We shall discuss here, however, five theories of failure, each of which has its limited use. These are :

1. Principal stress theory
2. Maximum shear stress theory
3. Principal strain theory
4. Total strain energy theory
5. Shear strain energy theory.

13.2. PRINCIPAL STRESS THEORY (RANKINE'S THEORY OR MAXIMUM NORMAL STRESS THEORY)

According to this theory, the failure occurs when the maximum principal stress in the compound stress system is equal to the yield stress in simple tension or compression test.

If σ_1 and σ_2 are the principal stresses in a 2-dimensional stress system and $\pm\sigma_0$ is the yield stress in simple tension or compression, then according to this theory the failure will occur when :

$$\sigma_1 = \pm\sigma \qquad ...(13.1)$$

$$\sigma_2 = \pm\sigma \qquad ...(13.2)$$

As the experimental evidence shows that brittle materials fail due to normal stresses, usually tensile, this theory may be used to predict failure in brittle materials. However, the brittle materials do not have a yield point. As such, instead of yield stress, ultimate stress σ_u is generally used as the failure criterion. It is common to use a factor of safety alongwith σ_u, since the ultimate failure in a brittle material occurs suddenly without warning. In this way, the failure equations become,

$$\left.\sigma_1 = \pm\frac{\sigma_u}{F.O.S.}\right\} \qquad ...(13.3)$$

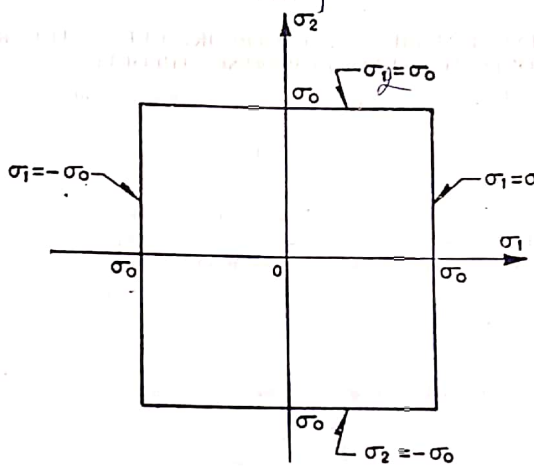


Fig. 13.1

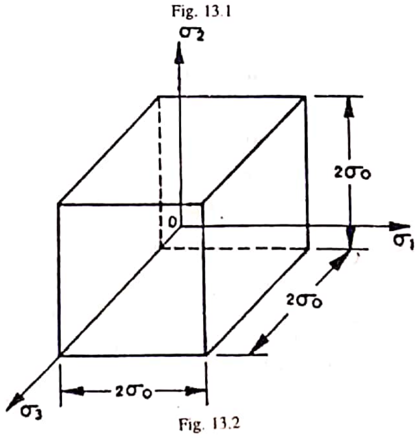


Fig. 13.2

$$\sigma_2 = \pm \frac{\sigma_u}{F.O.S.} \quad ...(13.4)$$

Yield locus is a square on $\sigma_1 - \sigma_2$ diagram as shown in Fig. 13.1. In three dimensional stress system, the yield surface is a cube with sides $2\sigma_0$ each with origin at the centroid of the cube (Fig. 13.2).

It must be mentioned here that this theory predicts failure at the same stress level irrespective of whether the stresses are tensile or compressive. This does not agree with practical results since brittle materials, *e.g.* cast iron, are much weaker in tension than in compression.

13.3. MAXIMUM SHEAR STRESS THEORY (GUEST'S THEORY OR COULOMB'S THEORY OR TRESCA THEORY)

In a 2-dimensional stress system, maximum shear stress may be any of the following :

$$\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1}{2}, \frac{\sigma_2}{2} \qquad (\because \sigma_3 = 0)$$

This theory states that the failure occurs when maximum shear stress induced in the compound stress system reaches the value of maximum shear stress in simple tension or compression test at yield point.

Now if $\pm\sigma_0$ is the yield stress, then the shear stress due to this

$$= \pm \frac{\sigma_0}{2}.$$

$\therefore$ According to this theory the failure occurs, when

$$\frac{\sigma_1 - \sigma_2}{2} = \pm \frac{\sigma_0}{2}$$

$$\frac{\sigma_1}{2} = \pm \frac{\sigma_0}{2}$$

or

$$\frac{\sigma_2}{2} = \pm \frac{\sigma_0}{2}$$

$\therefore$

$$\sigma_1 - \sigma_2 = \pm \sigma_0 \quad ...(13.5)$$

$$\sigma_1 = \pm \sigma_0 \quad ...(13.6)$$

$$\sigma_2 = \pm \sigma_0 \quad ...(13.7)$$

All the above equations can be written as,

$$|(\sigma_1 - \sigma_2)| = \sigma_0 \quad ...(13.8)$$

The yield locus is shown in the Fig. 13.3, which is commonly called the Tresca hexagon. In three-dimensional stress system, the yield surface is a hexagonal prism with open ends. The axis of the prism is equally inclined to

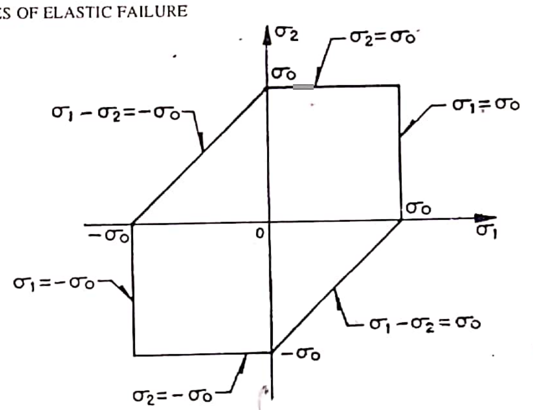


Fig. 13.3

each of the axes σ_1, σ_2 and σ_3. Looking in the direction of the prism axis the view is a regular hexagon of each side equal to $\sqrt{\frac{2}{3}}\,\sigma_0$

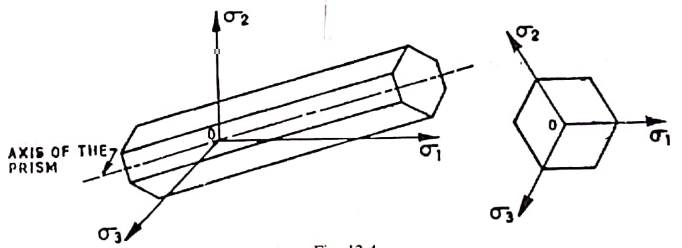


Fig. 13.4

It may be seen from the equation governing this criterion, *viz.* : $|\pm(\sigma_1 - \sigma_2)| = \sigma_0$, that adding the same normal stress to σ_1 and σ_2 will not affect the response of the material. In other words, on adding of hydrostatic stresses, tensile or compressive, no change in material response is predicted by this theory. This observation is true in case of ductile materials.

Maximum shear stress theory has been found to give conservative and hence safe results for ductile materials. Due to this reason and also because of the simple governing equation, the theory is widely used for ductile materials.

13.4. PRINCIPAL STRAIN THEORY (OR ST. VENANT'S THEORY OR MAXIMUM NORMAL STRAIN THEORY)

This states that the failure by yielding will occur when maximum strain in the compound stress system reaches the value of maximum strain at yield point in the simple tension or compression test.

Strain in the direction of $\sigma_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$ (ν = Poisson's ratio)

$$= \frac{\sigma_1}{E} - \frac{\sigma_2}{mE} \qquad \left(m = \frac{1}{\nu}\right)$$

Strain in the direction of $\sigma_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E}$

$$= \frac{\sigma_2}{E} - \frac{\sigma_1}{mE}$$

If $\pm\sigma_0$ is the yield point stress in simple tension or compression test, the maximum strain corresponding to this $= \pm\frac{\sigma_0}{E}$

According to this theory,

$$\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \pm\frac{\sigma_0}{E}$$

$$\frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} = \pm\frac{\sigma_0}{E}$$

$$\therefore \quad \sigma_1 - \nu\sigma_2 = \pm\sigma_0 \qquad \text{...(13.9)}$$

$$\sigma_2 - \nu\sigma_1 = \pm\sigma_0 \qquad \text{...(13.10)}$$

The experimental evidence does not support this theory and hence it is no more in practical use.

Plotting the above equations, we get the yield locus as shown in Fig. 13.5.

In 3-dimensional stress system the yield surface is given by the six faces of two, three-sided straight pyramids, placed back to back in inverted positions.

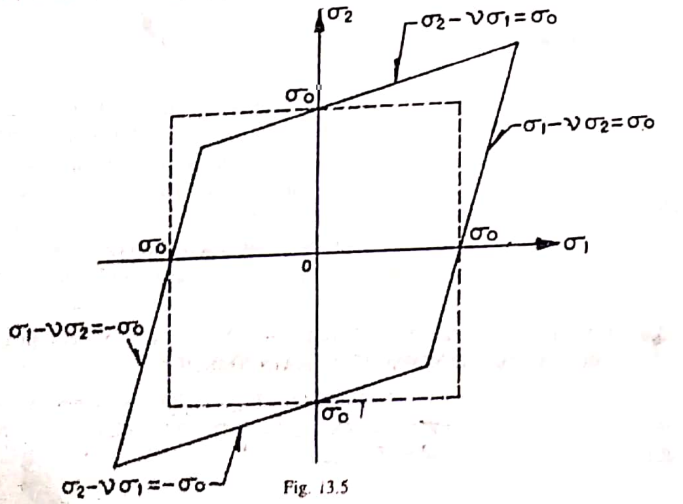


Fig. 13.5

The axis of the pyramids is equally inclined to the principal axes. The sections of the yield surface normal to the axis are equilateral triangles. Like principal stress theory, the yield surface ~~in this is thus~~ also bounded.

13.5. TOTAL STRAIN ENERGY THEORY (OR BELTRAMI OR HUBER OR HEIGH THEORY)

According to this theory, the failure will occur when the total strain energy density in the compound stress system is equal to the strain energy density at yield point in simple tension or compression test.

If σ_1 and σ_2 are the principal stresses, total S.E. density (*i.e.* strain energy per unity volume) is given as

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2] \qquad \text{...(a)}$$

If $\pm\sigma_0$ is the yield stress in simple tension or compression test, then strain

$$\text{energy density} = \frac{\sigma_0^2}{2E} \qquad \text{...(b)}$$

According to this theory, (*a*) and (*b*) should be equal.

i.e. $$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 \qquad \text{...(13.11)}$$

Plotting the above equation (and taking $\nu = 0.3$) we get the yield locus as ellipse (Fig. 13.6). The major and minor axes of the ellipse are at 45° and 135° respectively to the σ_1 axis.

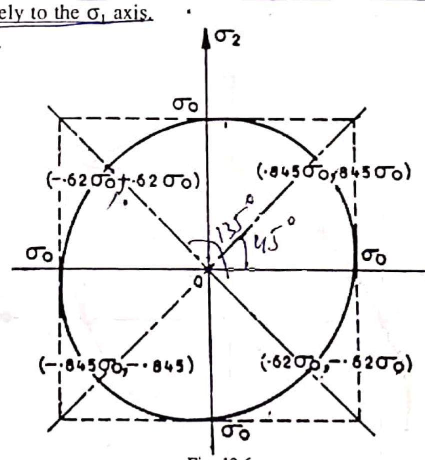


Fig. 13.6

13.6. SHEAR S.E. THEORY (DISTORTION ENERGY THEORY OR MISES-HENCKY OR VON MISES THEORY)

According to this theory, the failure will occur when the shear S.E. density under the compound stress system is equal to the shear S.E. density at yield point in simple tension or compression test.

Shear S.E. density in the general compound stress system,

$$u_s = \frac{1+\nu}{3E}\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]$$

$$= \frac{1+\nu}{6E}\,[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \qquad ...(13.12)$$

Shear S.E. density at Y.P. in simple tension or comp. test

$$= \frac{1+\nu}{3E}\,\sigma_0^2 \qquad ...(13.13)$$

Shear S.E. density in biaxial compound stress system

$$= \frac{1+\nu}{3E}\,[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2] \qquad ...(13.14)$$

According to this theory, Eq. (13.13) = Eq. (13.14)

i.e.,
$$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \qquad ...(13.15)$$

The same result may be expressed alternatively by means of a von Mises *equivalent stress* or *effective stress* defined as

$$\sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \qquad ...(13.16)$$

Then failure would occur when this equivalent stress (which is a normal stress) is equal to the yield stress, σ_0

On plotting Eq. (13.15) on $\sigma_1 - \sigma_2$ axes, we get yield locus as an ellipse as shown in Fig. 13.7.

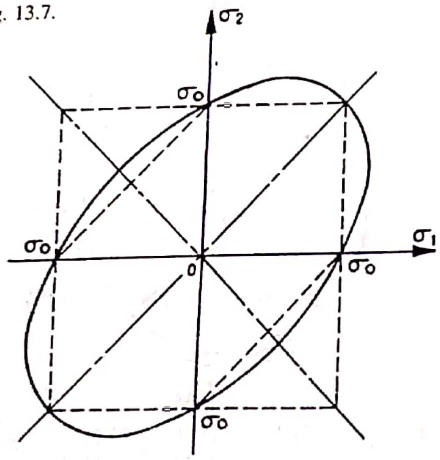


Fig. 13.7

For three-dimensional stress system, yield surface is an open right circular cylinder equally inclined to the σ_1, σ_2 and σ_3 axes (Fig. 13.7). The cross-sections

FRIDAY-HQ

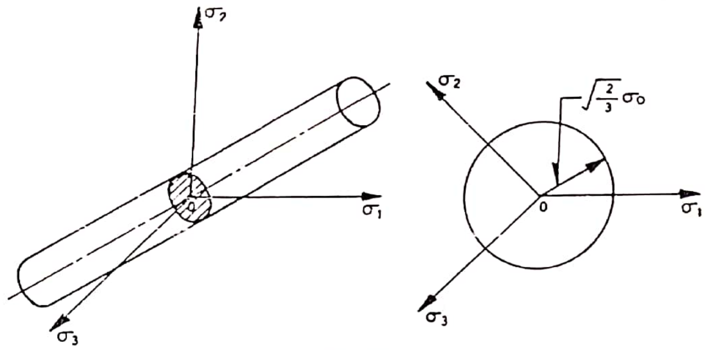


Fig. 13.8

of the cylinder looking in the direction of its axis are circles of radius equal to $\sqrt{\frac{2}{3}}\,\sigma_0$.

As can be seen from the yield locii, this theory agrees fairly well with the maximum shear stress theory. The hexagon representing the later is enclosed by the ellipse representing the former. Thus the later is more conservative.

This theory has been found to agree best with the experimental results on ductile materials and for that reason it is extensively used in design.

Note. The octahedral shear stress is the shear stress on octahedral planes. These are the planes equally inclined to the principal planes. This stress is given by,

$$\tau_{oct} = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad ...(13.17)$$

$\therefore$ For uniaxial loading, at yield point,

$$\tau_{oct} = \frac{\sqrt{2}}{3}\,\sigma_0 \qquad ...(13.18)$$

and for biaxial stress system,

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \qquad ...(13.19)$$

$\therefore$ Equating Eqs. (13.17) and (13.18) or Eqs. (13.18) and (13.19), we get the same results as obtained earlier in the case of shear S.E. theory.

Thus this theory called, *'Octahedral shear stress thory'* gives exactly the same failure criterion as given by 'shear S.E. theory'.

Example 13.1. *A thin cylindrical pressure vessel of diameter d and wall thickness t is made out of a material whose yield stress is σ_0. If the vessel is subjected to internal pressure p, determine the value of p at which yielding of vessel material will take place. Use von Mises criterion of failure.*

Sol. The hoop stress and the longitudinal stress respectively are given by,

$$\sigma_1 = \frac{pd}{2t}$$

and $$\sigma_2 = \frac{pd}{4t}$$

$\therefore$ von Mises equivalent stress, from Eq. (13.16),

$$\sigma_e = \sqrt{\left(\frac{pd}{2t}\right)^2 + \left(\frac{pd}{4t}\right)^2 - \left(\frac{pd}{2t}\right)\left(\frac{pd}{4t}\right)}$$

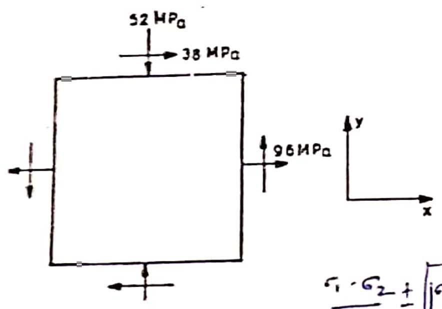
$$= \frac{\sqrt{3}\, pd}{4t}$$

For yielding to take place, $\sigma_e = \sigma_0$

i.e. $$\sqrt{3}\,\frac{pd}{4t} = \sigma_0$$

$\therefore$ $$p = \frac{4t\,\sigma_0}{\sqrt{3}d}$$

Example 13.2. *At a point in a steel structure, the state of plane stress occurring is shown in Fig. 13.9. Determine the safety factors according to (i) Tresca (ii) Von Mises criteria. Yield point for steel is 320 MPa.*



$$\frac{\sigma_1 - \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

Fig. 13.9

Sol. Principal stresses are,

$$\sigma_{1,2} = \frac{96 - 52}{2} \pm \sqrt{\left\{\frac{96 - (-52)}{2}\right\}^2 + (38)^2}$$

$$= 22 \pm 83.2$$

$$= 105.2 \text{ MPa},\ -61.2 \text{ MPa}$$

$\therefore$ Absolute maximum shear stress

$$= \frac{105.2 - (-61.2)}{2} = 83.2 \text{ MPa}$$

Also from Eq. (13.16),

Von Mises equivalent stress,

$$\sigma_e = \sqrt{(105.2)^2 + (-61.2)^2 - (105.2)(-61.2)}$$

$$= 145.8 \text{ MPa}$$

$\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$

(*i*) *Tresca criterion*

Factor of safety $$= \frac{320/2}{83.2} = \mathbf{1.92}$$

(*ii*) *Von Mises criterion*

Factor of safety $$= \frac{320}{145.8} = \mathbf{2.19}$$

Example 13.3. *A steel bolt is subjected to a direct pull of 20 kN and transverse shear force of 10 kN. Calculate the diameter of the bolt using all theories of failure. Take yield point stress for steel as 250 MPa and factor of safety as 2. Poisson's ratio may be taken as 0.3.*

Sol. Safe value of $$\sigma_0 = \frac{250}{2} = 125 \text{ N/mm}^2$$

Let A = Cross-sectional area of the bolt required, mm^2

Then $$\sigma_x = \frac{20 \times 10^3}{A}$$

$$\tau = \frac{10 \times 10^3}{A}$$

$\therefore$ $$\sigma_1, \sigma_2 = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau^2}$$

$$= \frac{20 \times 10^3}{2A} \pm \sqrt{\left(\frac{20 \times 10^3}{2A}\right)^2 + \left(\frac{10 \times 10^3}{A}\right)^2}$$

$$= \frac{10^4}{A}\,[1 \pm \sqrt{2}]$$

$$\sigma_1 = \frac{2.414 \times 10^4}{A} \text{ N/mm}^2$$

$$\sigma_2 = \frac{-.414 \times 10^4}{A} \text{ N/mm}^2$$

1. *Principal stress theory*

$$\frac{2.414 \times 10^4}{A} = 125$$